

## Theory of Structure (315313)

MSBTE K-Scheme · Civil Engineering (CE) Semester 5 · Model Question Paper — Set C · Model Answers

Sign conventions used: sagging BM positive (+), hogging BM negative (–) in diagrams; support moments in three-moment equations are written as hogging magnitudes; in moment distribution, clockwise end moments are positive. Units: kN, m unless stated.

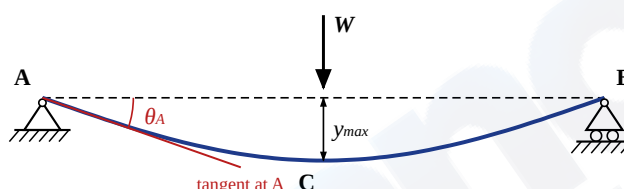
### Q.1 Attempt any FIVE of the following: (10 M)

#### a) Slope and deflection with units

2 M

**Slope:** The slope at any section of a deflected beam is the **angle (in radians) made by the tangent to the elastic curve** at that section with the original (undeflected) axis of the beam.  $\theta = dy/dx$ . Unit: radian.

**Deflection:** The **vertical displacement** of any point on the axis of the beam from its original position, measured perpendicular to the original axis, is called deflection ( $y$ ). Unit: mm (or m).



Dashed: original axis · Blue: elastic curve (deflected axis)

Fig.: Slope ( $\theta$ ) and deflection ( $y$ ) of a simply supported beam

(Each definition ½ M, sketch 1 M)

#### b) Differential equation of elastic curve

2 M

**$EI \frac{d^2y}{dx^2} = M$** , where  $E$  = modulus of elasticity,  $I$  = moment of inertia of the section,  $EI$  = flexural rigidity,  $y$  = deflection at distance  $x$ ,  $M$  = bending moment at the section. If  $y$  is the deflection at a section at distance  $x$ , then for small deflections:

- Deflection =  $y$
- Slope =  $\frac{dy}{dx} = \theta$
- Bending moment  $M = EI \frac{d^2y}{dx^2}$
- Shear force  $F = EI \frac{d^3y}{dx^3}$
- Rate of loading  $w = EI \frac{d^4y}{dx^4}$

Also from the flexure formula  $M/I = E/R$ , so  $1/R = M/EI = d^2y/dx^2$  ( $R$  = radius of curvature).

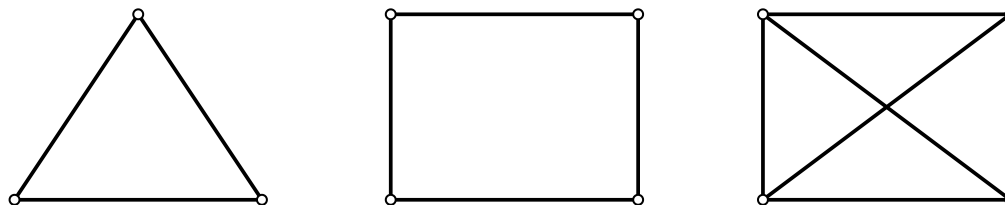
(Any two correct relations, 1 M each)

#### c) Deficient and redundant frames

2 M

**Deficient frame:** An imperfect frame having **fewer members than required** for a perfect frame, i.e.  $m < 2j - 3$ . It cannot resist loads and collapses (unstable).

**Redundant frame:** An imperfect frame having **more members than required**, i.e.  $m > 2j - 3$ . It is statically indeterminate; member forces cannot be found by equilibrium equations alone.



(a) Perfect:  $m = 3, j = 3, 2j - 3 = 3$  (b) Deficient:  $m = 4 < 2j - 3 = 5$  (c) Redundant:  $m = 6 > 2j - 3 = 5$

(Each definition 1 M)

#### d) Stiffness factor and carry-over factor

2 M

**Carry-over factor:** Ratio of the moment induced at the fixed far end to the moment applied at the near end of a member. For a prismatic member with far end fixed it is  $\frac{1}{2}$ ; with far end hinged it is 0.

**Stiffness factor:** Moment required at the near end to rotate it through unit angle (1 radian) —  $k = 4EI/L$  (far end fixed),  $3EI/L$  (far end hinged).

(Each 1 M)

#### e) Advantages of fixed beam

2 M

1. The **maximum bending moment is smaller** than in a SS beam of the same span and loading ( $wL^2/12$  against  $wL^2/8$  for UDL), so a smaller, economical section is needed.
2. The **deflection is much less** ( $wL^4/384EI$  against  $5wL^4/384EI$  — one-fifth for UDL), and slope at the ends is zero, so the beam is stiffer.

(Any two advantages, 1 M each)

#### f) Assumptions in analysis of trusses

2 M

1. All members are **pin-jointed** (hinged) at their ends; joints are frictionless.
2. **Loads act only at the joints** (nodes).
3. The **self-weight of members is neglected**.

(Any two, 1 M each)

#### g) Sign convention in moment distribution

2 M

In the moment distribution method (Hardy Cross), the moment acting on the end of a member is taken **positive when clockwise** and **negative when anticlockwise**. Thus for a fixed beam with UDL,  $FEM_{AB} = -wL^2/12$  and  $FEM_{BA} = +wL^2/12$ .

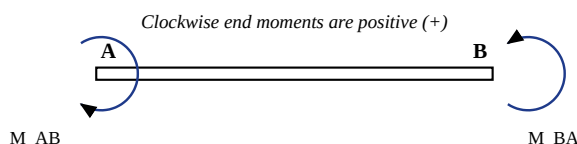


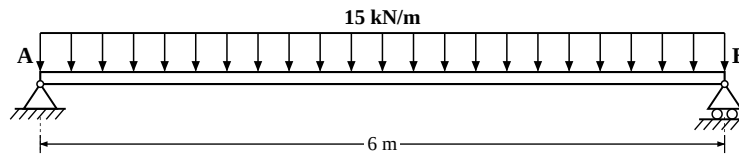
Fig.: Sign convention for member end moments in moment distribution

(Convention 1 M, example/sketch 1 M)

## Q.2 Attempt any THREE of the following: (12 M)

### a) Required M.I. from permissible deflection

4 M



**Given:**  $L = 6 \text{ m} = 6000 \text{ mm}$ ,  $w = 15 \text{ kN/m} = 15 \text{ N/mm}$ ,  $E = 200,000 \text{ N/mm}^2$ , permissible deflection  $= L/325$

Permissible deflection  $y = 6000 / 325 = \mathbf{18.46 \text{ mm}}$

For a SS beam with UDL, maximum deflection  $y_{\max} = 5wL^4 / 384EI$

$$18.46 = 5 \times 15 \times 6000^4 / (384 \times 200,000 \times I)$$

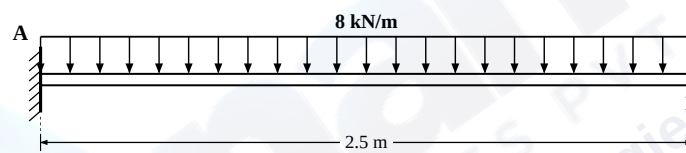
$$I = 5 \times 15 \times 6000^4 / (384 \times 200,000 \times 18.46) = \mathbf{6.855 \times 10^7 \text{ mm}^4}$$

$$\text{Maximum slope with this } I: \theta = wL^3 / 24EI = 15 \times 6^3 / (24 \times 13710.94) = \mathbf{0.00985 \text{ rad}}$$

(Permissible deflection 1 M, formula 1 M, I 1 M, slope 1 M)

### b) Timber cantilever with UDL

4 M



Cantilever AB (A fixed, B free)

**Given:**  $L = 2.5 \text{ m}$ ,  $w = 8 \text{ kN/m}$ ,  $E = 12,000 \text{ N/mm}^2$

$$I = bd^3/12 = 120 \times 250^3 / 12 = \mathbf{1.562 \times 10^8 \text{ mm}^4}$$

$$EI = 12,000 \times 1.562 \times 10^8 = 1.875 \times 10^{12} \text{ N}\cdot\text{mm}^2 = \mathbf{1875 \text{ kN}\cdot\text{m}^2}$$

| Load case           | Slope at B | $\theta$ (rad) | Deflection at B | y (mm) |
|---------------------|------------|----------------|-----------------|--------|
| UDL w on whole span | $wL^3/6EI$ | 0.01111        | $wL^4/8EI$      | 20.83  |

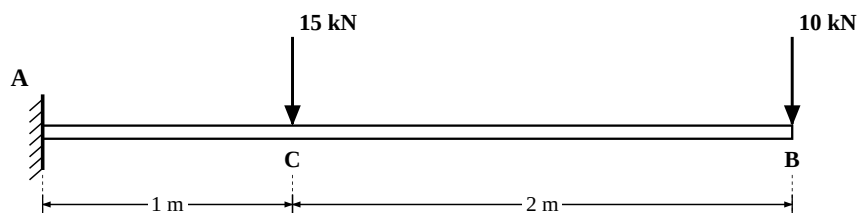
Slope at free end  $\theta_B = \mathbf{0.01111 \text{ rad}}$  ( $0.637^\circ$ )

Deflection at free end  $y_B = \mathbf{20.83 \text{ mm}}$  (downward)

(I and EI 1 M, formulae 1 M, slope 1 M, deflection 1 M)

### c) Macaulay's method — cantilever

4 M



**Macaulay's method** — take the **free end B as origin** and section X-X at distance  $x$  from B in the last segment (towards the fixed end A). Bending moment of a cantilever is hogging (negative).

$$EI \frac{d^2y}{dx^2} = M_x = -10x \mid -15[x - 2]$$

$$EI \frac{dy}{dx} = -5x^2 \mid -7.5[x - 2]^2 + C_1 \dots (1)$$

$$EI y = -1.667x^3 \mid -2.5[x - 2]^3 + C_1x + C_2 \dots (2)$$

**Boundary conditions at the fixed end A ( $x = 3$  m):**  $dy/dx = 0 \rightarrow C_1 = 52.5$ ;  $y = 0 \rightarrow C_2 = -110$

- At B ( $x = 0$  m from free end):  $EI \theta = 52.5 \rightarrow \theta_B = 52.5/EI$ ;  $EI y = -110 \rightarrow y_B = 110/EI$  (downward)
- At C ( $x = 2$  m from free end):  $EI \theta = 32.5 \rightarrow \theta_C = 32.5/EI$ ;  $EI y = -18.333 \rightarrow y_C = 18.33/EI$  (downward)

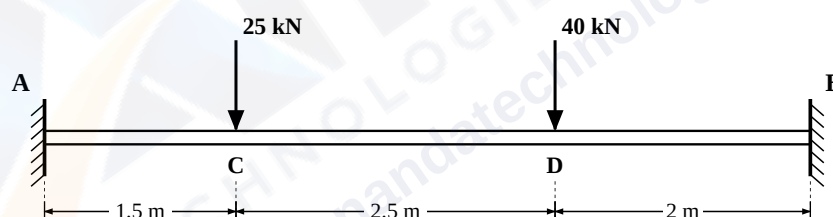
For a cantilever, the maximum slope and maximum deflection occur at the **free end**.

✓ Check: slope and deflection agree with an independent stiffness-matrix analysis.

(Equation 1 M, constants 1 M, values at B and C 2 M)

### d) Fixed beam — FEM and BMD

4 M



**Step 1 — Fixed end moments** (hogging, by standard formulae):

| Load                                                       | $M_A$ (kN·m)                                         | $M_B$ (kN·m)                                        |
|------------------------------------------------------------|------------------------------------------------------|-----------------------------------------------------|
| Point load 25 kN at 1.5 m from A ( $a = 1.5$ , $b = 4.5$ ) | $Wab^2/L^2 = 25 \times 1.5 \times 4.5^2/6^2 = 21.09$ | $Wa^2b/L^2 = 25 \times 1.5^2 \times 4.5/6^2 = 7.03$ |
| Point load 40 kN at 4 m from A ( $a = 4$ , $b = 2$ )       | $Wab^2/L^2 = 40 \times 4 \times 2^2/6^2 = 17.78$     | $Wa^2b/L^2 = 40 \times 4^2 \times 2/6^2 = 35.56$    |
| <b>Total</b>                                               | <b>38.87</b>                                         | <b>42.59</b>                                        |

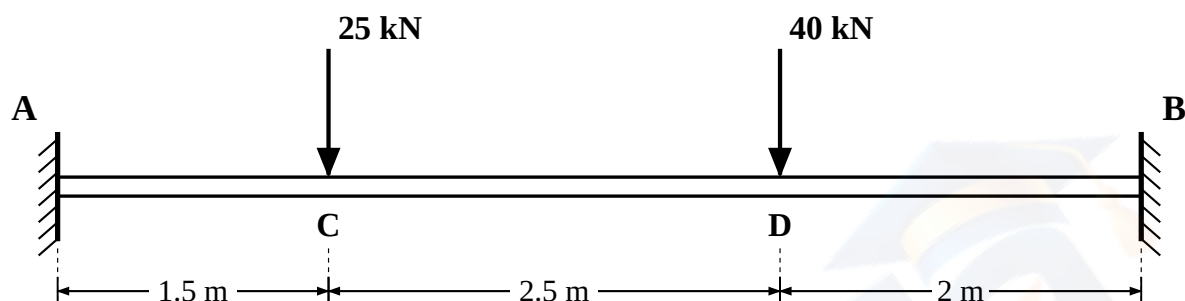
Fixed end moments:  $M_A = -38.87$  kN·m and  $M_B = -42.59$  kN·m (negative = hogging).

**Step 2 — Reactions:**  $R_A = (\text{free reaction}) + (M_A - M_B)/L = 32.08 + (38.87 - 42.59)/6 = 31.46$  kN;  $R_B = 32.92 - (38.87 - 42.59)/6 = 33.54$  kN

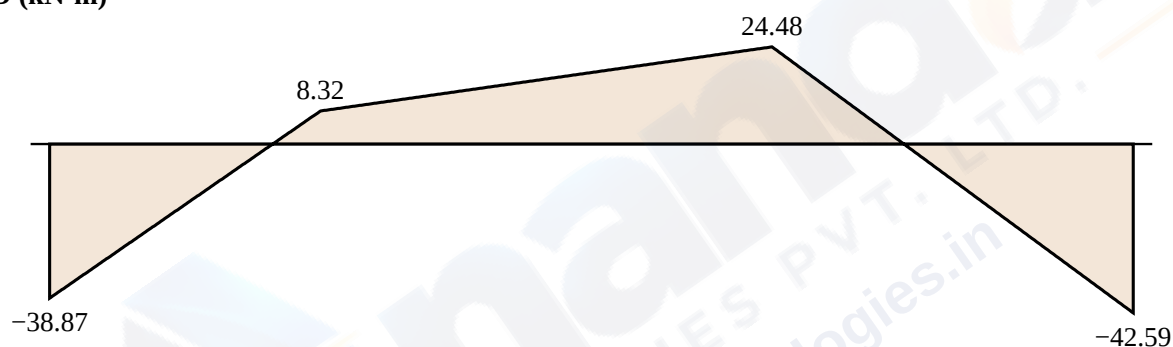
**Step 3 — Bending moments** (net BM = free BM of SS beam – fixed end moment line):

| Section (from A) | Free BM (kN·m) | Fixed-moment line (kN·m) | Net BM (kN·m) |
|------------------|----------------|--------------------------|---------------|
| A (0 m)          | 0              | 38.87                    | –38.87        |
| 1.5 m            | 48.12          | 39.8                     | 8.32          |
| 4 m              | 65.83          | 41.35                    | 24.48         |
| B (6 m)          | 0              | 42.59                    | –42.59        |

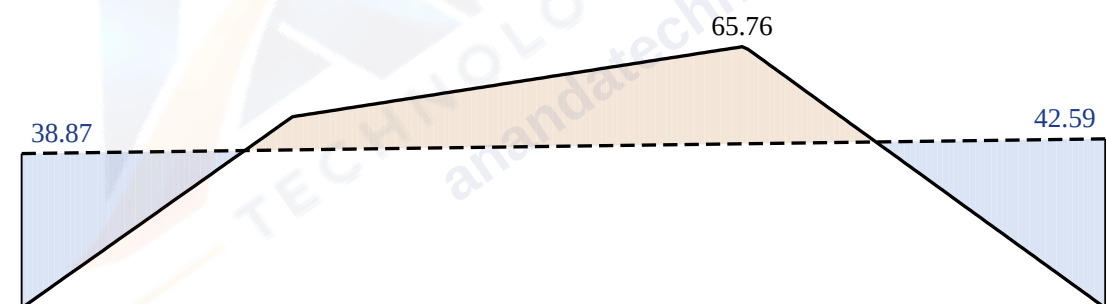
Points of contraflexure (BM = 0): **1.24 m, 4.73 m** from A.



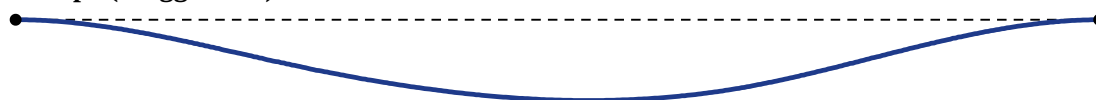
**BMD (kN·m)**



**BMD by superposition (kN·m): free BMD (solid) and support moment line (dashed)**



**Deflected shape (exaggerated)**



*Fixed beam — net BMD, BMD by superposition and deflected shape*

✓ Check: stiffness-matrix analysis gives the same end moments (38.87, 42.59 kN·m).

(FEMs 2 M, BM values 1 M, BMD 1 M)

### Q.3 Attempt any THREE of the following: (12 M)

#### a) Fixed beam vs SS beam

4 M

| Fixed beam                                                                       | Simply supported beam                              |
|----------------------------------------------------------------------------------|----------------------------------------------------|
| Both ends fixed; slope and deflection zero at ends.                              | Ends free to rotate; only deflection zero at ends. |
| Statically indeterminate.                                                        | Statically determinate.                            |
| Max BM for UDL = $wL^2/12$ (at supports).                                        | Max BM for UDL = $wL^2/8$ (at mid-span).           |
| Max deflection for UDL = $wL^4/384EI$ .                                          | Max deflection for UDL = $5wL^4/384EI$ .           |
| BMD has hogging at ends and sagging at centre, with two points of contraflexure. | BMD is only sagging, no point of contraflexure.    |

(Any four points, 1 M each)

#### b) Clapeyron's theorem and fixed end

4 M

**Statement:** For a continuous beam with two consecutive spans  $L_1$  and  $L_2$  supported at A, B and C (at the same level), the support moments  $M_A$ ,  $M_B$  and  $M_C$  are related as

$$M_A L_1 + 2M_B(L_1 + L_2) + M_C L_2 = 6A_1 \bar{x}_1 / L_1 + 6A_2 \bar{x}_2 / L_2$$

where  $A_1, A_2$  = areas of the free BMDs of spans AB and BC (each treated as simply supported);  $\bar{x}_1$  = distance of centroid of  $A_1$  from the left end A;  $\bar{x}_2$  = distance of centroid of  $A_2$  from the right end C. Support moments are hogging. If the M.I. differs, each  $L$  is replaced by  $L/I$  and each  $6A\bar{x}/L$  by  $6A\bar{x}/(LI)$ .

**Explanation / use:** The equation is written for every set of three consecutive supports; the simultaneous equations give the unknown support moments. For a simply supported end the moment is zero; for a fixed end an imaginary span of zero length is added. Standard values: UDL  $\rightarrow 6A\bar{x}/L = wL^3/4$ ; central point load  $\rightarrow 6A\bar{x}/L = 3WL^2/8$ .

Free BMDs (areas  $A_1, A_2$ ) and support moment line (dashed)

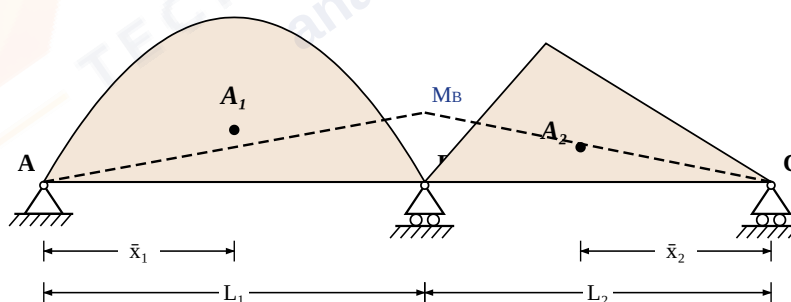


Fig.: Two consecutive spans for Clapeyron's theorem of three moments

**Fixed end:** If end A is fixed, an imaginary span  $A_0A$  of zero length ( $L_0 = 0$ ) with a simple support  $A_0$  ( $M_{A_0} = 0$ ) is assumed. Applying the theorem to  $A_0, A, B$  gives  $2M_A L_1 + M_B L_1 = 6A_1 \bar{x}_1' / L_1$  ( $\bar{x}_1'$  measured from B).

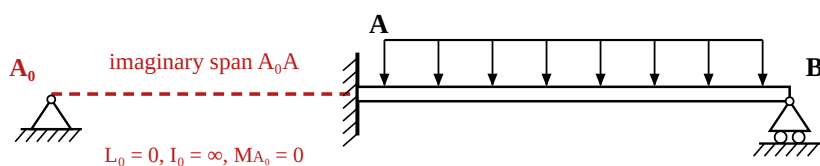
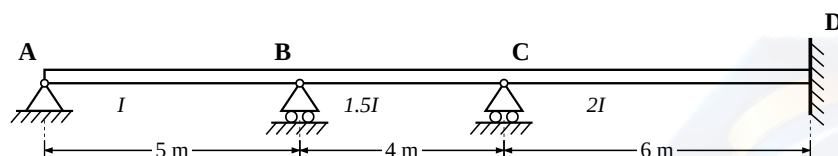


Fig.: Fixed end A replaced by a simple support  $A_0$  with an imaginary span of zero length

(Statement with equation 2 M, explanation 1 M, fixed end 1 M)

### c) Distribution factors for beam ABCD

4 M



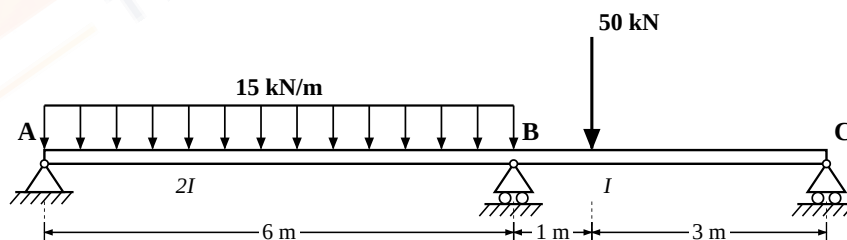
Relative stiffness  $k = I/L$  when the far end is fixed or continuous. Member AB has its far end A simply supported, so at B,  $k_{BA} = (3/4) \cdot I/L$ . At the simple end A,  $DF = 1$ ; at the fixed end D,  $DF = 0$ .

| Joint | Member | Relative stiffness $k$    | $\Sigma k$ | Distribution factor     |
|-------|--------|---------------------------|------------|-------------------------|
| A     | AB     | $I/5 = 0.2I$              | $0.2I$     | 1 (simple end)          |
| B     | BA     | $(3/4) \cdot I/5 = 0.15I$ | $0.525I$   | $0.15/0.525 = 0.286$    |
|       | BC     | $1.5I/4 = 0.375I$         |            | $0.375/0.525 = 0.714$   |
| C     | CB     | $1.5I/4 = 0.375I$         | $0.7083I$  | $0.375/0.7083 = 0.529$  |
|       | CD     | $2I/6 = 0.3333I$          |            | $0.3333/0.7083 = 0.471$ |
| D     | DC     | $2I/6 = 0.3333I$          | $0.3333I$  | 0 (fixed end)           |

(Stiffnesses 2 M, DFs 2 M)

### d) Three moment theorem — different M.I.

4 M



**Step 1 — Free bending moments and  $6A\bar{x}/L$  terms** (each span treated as simply supported). For UDL:  $6A\bar{x}/L = wL^3/4$ ; for point load  $W$  at distance  $a$  from the left end:  $6A\bar{x}/L = Wab(L + a)/L$  about the left end and  $Wab(L + b)/L$  about the right end.

| Span | L (m) | Max free BM (kN·m) | Area A | $\bar{x}$ from left (m) | $\bar{x}$ from right (m) | $6A\bar{x}_{\text{left}}/L$ | $6A\bar{x}_{\text{right}}/L$ |
|------|-------|--------------------|--------|-------------------------|--------------------------|-----------------------------|------------------------------|
|------|-------|--------------------|--------|-------------------------|--------------------------|-----------------------------|------------------------------|

|    |   |                                            |     |      |      |       |       |
|----|---|--------------------------------------------|-----|------|------|-------|-------|
| AB | 6 | $wL^2/8 = 15 \times 6^2/8 = 67.5$          | 270 | 3    | 3    | 810   | 810   |
| BC | 4 | $W_{ab}/L = 50 \times 1 \times 3/4 = 37.5$ | 75  | 1.67 | 2.33 | 187.5 | 262.5 |

**Step 2 — Clapeyron's equations** (support moments  $M$  taken hogging positive; terms divided by  $I$  of each span):

$$M_1 L_1 / I_1 + 2M_2 (L_1 / I_1 + L_2 / I_2) + M_3 L_2 / I_2 = 6A_1 \bar{x}_1 / (L_1 I_1) + 6A_2 \bar{x}_2 / (L_2 I_2)$$

( $\bar{x}_1$  measured from the left support of the left span,  $\bar{x}_2$  from the right support of the right span.)

End A is simply supported, so  $M_A = 0$ . End C is simply supported, so  $M_C = 0$ .

For supports **A, B, C**:  $14 M_B = 667.5 \dots (1)$

Solving:  **$M_B = 47.68 \text{ kN}\cdot\text{m}$**  (hogging).

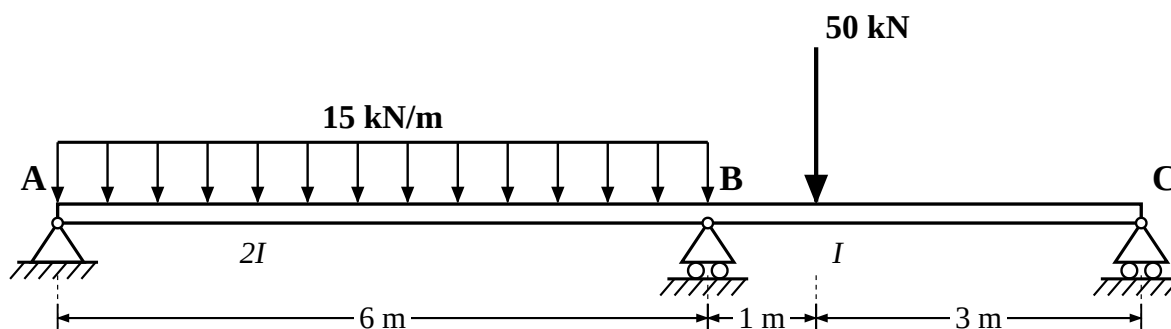
### Step 3 — Reactions, SFD and BMD

**Support reactions** (span-wise: free reaction  $\pm (M_{\text{left}} - M_{\text{right}})/L$ ):

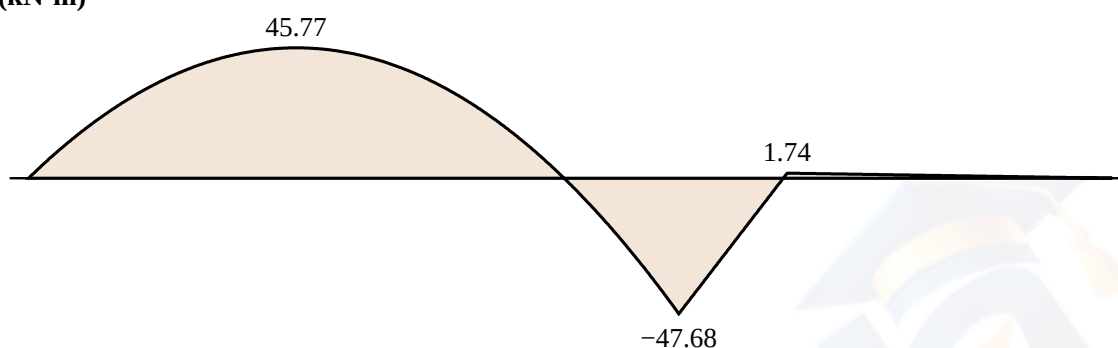
| Span     | Free reactions (left/right) kN | Correction (kN)         | Left end (kN) | Right end (kN) |
|----------|--------------------------------|-------------------------|---------------|----------------|
| AB (6 m) | 45 / 45                        | $(0 - 47.68)/6 = -7.95$ | 37.05         | 52.95          |
| BC (4 m) | 37.5 / 12.5                    | $(47.68 - 0)/4 = 11.92$ | 49.42         | 0.58           |

**$R_A = 37.05 \text{ kN}$ ;  $R_B = 102.37 \text{ kN}$ ;  $R_C = 0.58 \text{ kN}$**  ( $\Sigma R = 140 \text{ kN} = \text{total load } \checkmark$ )

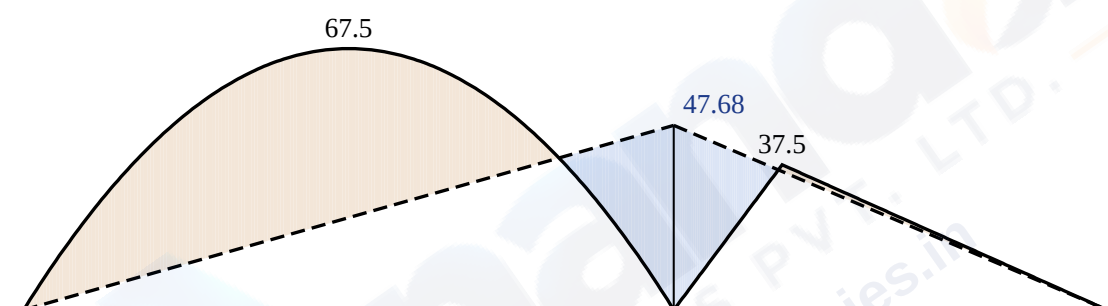
Maximum sagging moments — span AB:  **$45.77 \text{ kN}\cdot\text{m}$**  at 2.47 m from A; span BC:  **$1.74 \text{ kN}\cdot\text{m}$**  at 1 m from B.



BMD ( $\text{kN}\cdot\text{m}$ )



BMD by superposition ( $\text{kN}\cdot\text{m}$ ): free BMD (solid) and support moment line (dashed)



BMD (net) and BMD by superposition (+ sagging, - hogging)

✓ Check: support moments and reactions agree with an independent stiffness-matrix (and moment distribution) solution.

( $6A\bar{x}/L$  terms 1 M, equation with I 2 M,  $M_B$  1 M)

**Q.4** Attempt any THREE of the following: (12 M)

**a) Method of joints vs method of sections**

4 M

| Method of joints                                                                 | Method of sections                                                                                                  |
|----------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------|
| Equilibrium of each joint is considered ( $\Sigma F_x = 0$ , $\Sigma F_y = 0$ ). | Equilibrium of a part of the frame cut by a section is considered ( $\Sigma F_x$ , $\Sigma F_y$ , $\Sigma M = 0$ ). |
| At a joint not more than two unknowns can be solved.                             | Section should cut not more than three members with unknown forces.                                                 |
| Used when forces in all members are required.                                    | Used when forces in a few selected members are required.                                                            |
| Forces are found joint by joint; an error is carried forward.                    | A member force is found directly, independent of others.                                                            |

|                                 |                                                                   |
|---------------------------------|-------------------------------------------------------------------|
| Uses only resolution of forces. | Mostly uses moments about a convenient point (method of moments). |
|---------------------------------|-------------------------------------------------------------------|

(Any four points, 1 M each)

### b) Fixed beam with central point load — first principles

4 M

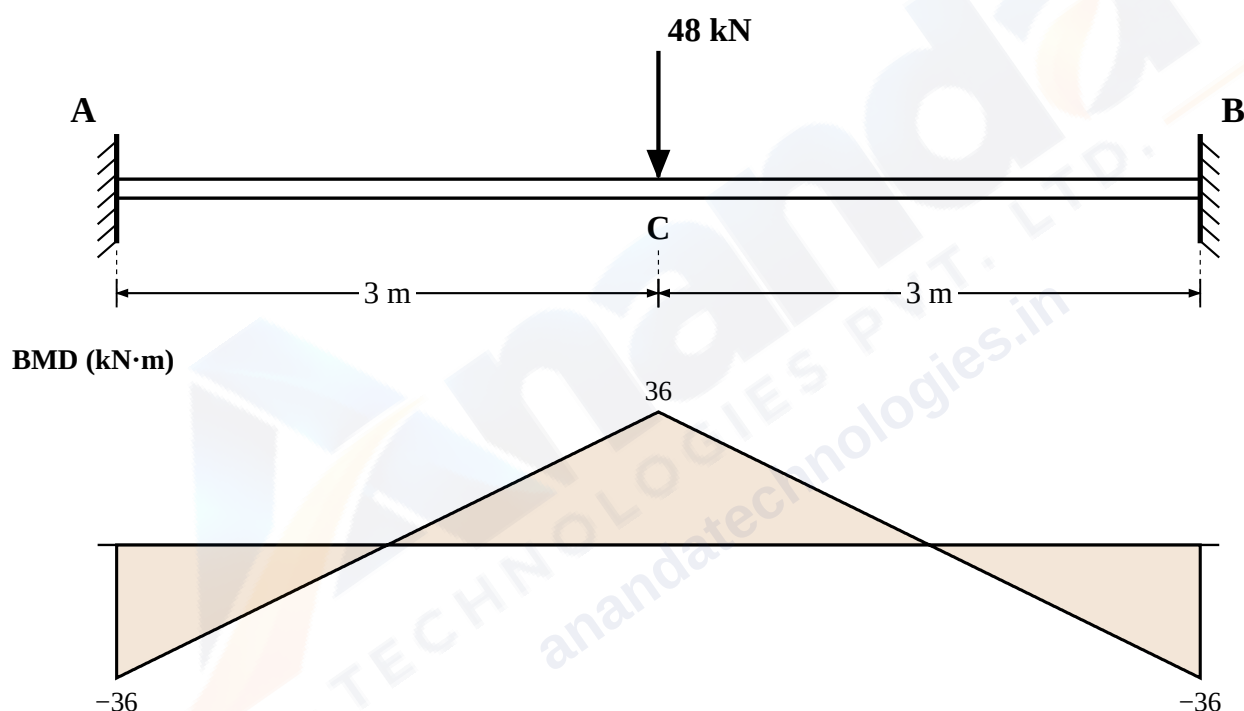
For a fixed beam, slope at both ends is zero, therefore (i) **area of free BMD = area of fixed-end moment diagram** and (ii) the centroids of the two diagrams lie on the same vertical.

For a central point load  $W$ , the free BMD is a triangle with maximum ordinate  $WL/4$ . Area =  $\frac{1}{2} \times L \times WL/4 = WL^2/8$ .

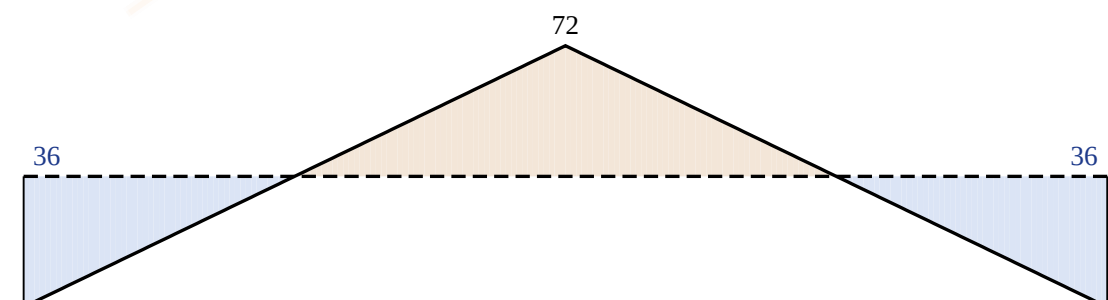
By symmetry  $M_A = M_B = M$ ; area of fixed-moment diagram =  $M \times L$ .

$M \times L = WL^2/8 \rightarrow M_A = M_B = WL/8$  (hogging). Net BM at centre =  $WL/4 - WL/8 = WL/8$  (sagging).

**Example:**  $L = 6$  m,  $W = 48$  kN  $\rightarrow M_A = M_B = 48 \times 6/8 = 36$  kN·m; BM at centre =  $36$  kN·m (sagging); points of contraflexure at  $L/4 = 1.5$  m from each end.



**BMD by superposition (kN·m): free BMD (solid) and support moment line (dashed)**



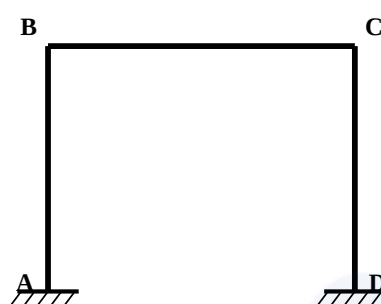
Fixed beam with central point load

(Principle 1 M, derivation 2 M, BMD 1 M)

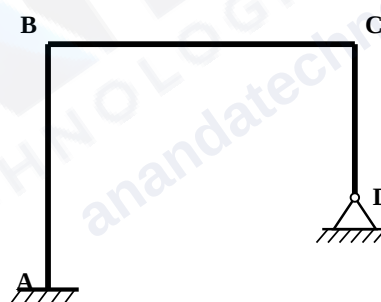
### c) Symmetrical vs unsymmetrical portal frames

4 M

| Symmetrical portal frame                                                       | Unsymmetrical portal frame                                    |
|--------------------------------------------------------------------------------|---------------------------------------------------------------|
| Frame and loading are symmetrical about the vertical centre line.              | Frame or loading (or both) is not symmetrical.                |
| Columns have equal height and equal M.I.; both bases have same end conditions. | Columns of unequal height / M.I. or different end conditions. |
| No sway (lateral displacement) occurs under vertical loads.                    | The frame sways sideways; sway moments must be considered.    |
| Analysis is simple (non-sway analysis; half frame can be used).                | Analysis is more complex (sway analysis required).            |



(a) Symmetrical portal frame: equal columns ( $h, I$ ), same end conditions, symmetrical load



(b) Unsymmetrical portal frame: unequal columns / different supports / unsymmetrical load — frame sways

(Any four points 1 M each (sketches included))

### d) Perfect, deficient and redundant frames

4 M

A **frame (truss)** is a structure made of members pin-jointed at their ends. Depending on the number of members ( $m$ ) and joints ( $j$ ):

1. **Perfect frame:**  $m = 2j - 3$ . Just enough members to keep the frame stable and in equilibrium; statically determinate.

2. **Imperfect frame:**  $m \neq 2j - 3$ . It is of two types:

- **Deficient frame:**  $m < 2j - 3$  — fewer members, frame is unstable and collapses under load.
- **Redundant frame:**  $m > 2j - 3$  — extra members, statically indeterminate.



(a) Perfect:  $m = 3, j = 3, 2j - 3 = 3$  (b) Deficient:  $m = 4 < 2j - 3 = 5$  (c) Redundant:  $m = 6 > 2j - 3 = 5$

(Definitions 2 M, sketches 2 M)

### e) Derivation of $M = EI \frac{d^2y}{dx^2}$

4 M

Consider a small portion of a beam bent into an arc of radius  $R$ . From the theory of simple bending,  $M/I = E/R$ , so  $1/R = M/(EI)$ .

For a curve  $y = f(x)$ , curvature  $1/R = (d^2y/dx^2) / [1 + (dy/dx)^2]^{3/2}$ . For a beam the slope  $dy/dx$  is very small, so  $(dy/dx)^2 \approx 0$  and  $1/R = d^2y/dx^2$ .

Equating:  $M = EI \frac{d^2y}{dx^2}$  — the differential equation of the elastic curve.

If  $y$  is the deflection at a section at distance  $x$ , then for small deflections:

- Deflection =  $y$
- Slope =  $dy/dx = \theta$
- Bending moment  $M = EI \frac{d^2y}{dx^2}$
- Shear force  $F = EI \frac{d^3y}{dx^3}$
- Rate of loading  $w = EI \frac{d^4y}{dx^4}$

Also from the flexure formula  $M/I = E/R$ , so  $1/R = M/EI = d^2y/dx^2$  ( $R$  = radius of curvature).

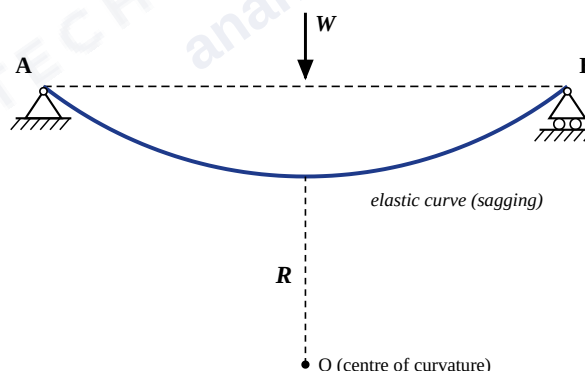


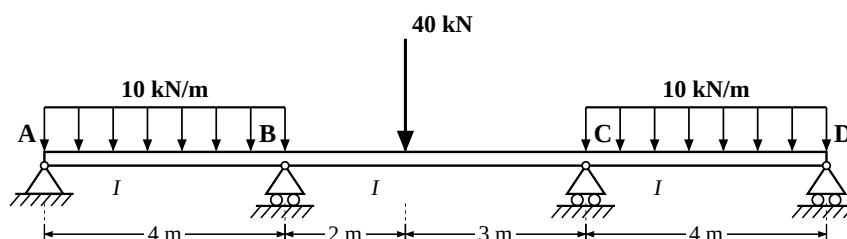
Fig.: Elastic curve of a SS beam with central point load and radius of curvature  $R$

(Derivation 2 M, relations 1 M, sketch 1 M)

## Q.5 Attempt any TWO of the following: (12 M)

### a) Three moment theorem — three-span beam

6 M



**Step 1 — Free bending moments and  $6A\bar{x}/L$  terms** (each span treated as simply supported). For UDL:  $6A\bar{x}/L = wL^3/4$ ; for point load  $W$  at distance  $a$  from the left end:  $6A\bar{x}/L = Wab(L + a)/L$  about the left end and  $Wab(L + b)/L$  about the right end.

| Span | L (m) | Max free BM (kN·m)                    | Area A | $\bar{x}$ from left (m) | $\bar{x}$ from right (m) | $6A\bar{x}_{\text{left}}/L$ | $6A\bar{x}_{\text{right}}/L$ |
|------|-------|---------------------------------------|--------|-------------------------|--------------------------|-----------------------------|------------------------------|
| AB   | 4     | $wL^2/8 = 10 \times 4^2/8 = 20$       | 53.33  | 2                       | 2                        | 160                         | 160                          |
| BC   | 5     | $Wab/L = 40 \times 2 \times 3/5 = 48$ | 120    | 2.33                    | 2.67                     | 336                         | 384                          |
| CD   | 4     | $wL^2/8 = 10 \times 4^2/8 = 20$       | 53.33  | 2                       | 2                        | 160                         | 160                          |

**Step 2 — Clapeyron's equations** (support moments  $M$  taken hogging positive):

$$M_1L_1 + 2M_2(L_1 + L_2) + M_3L_2 = 6A_1\bar{x}_1/L_1 + 6A_2\bar{x}_2/L_2$$

( $\bar{x}_1$  measured from the left support of the left span,  $\bar{x}_2$  from the right support of the right span.)

End A is simply supported, so  $M_A = 0$ . End D is simply supported, so  $M_D = 0$ .

For supports A, B, C:  $18 M_B + 5 M_C = 544 \dots (1)$

For supports B, C, D:  $5 M_B + 18 M_C = 496 \dots (2)$

Solving the simultaneous equations:  $M_B = 24.45 \text{ kN}\cdot\text{m}$ ,  $M_C = 20.76 \text{ kN}\cdot\text{m}$  (hogging).

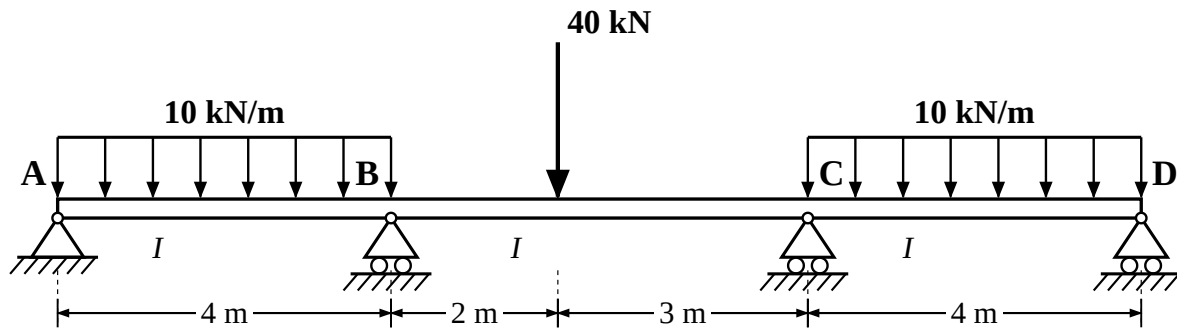
### Step 3 — Reactions, SFD and BMD

**Support reactions** (span-wise: free reaction  $\pm (M_{\text{left}} - M_{\text{right}})/L$ ):

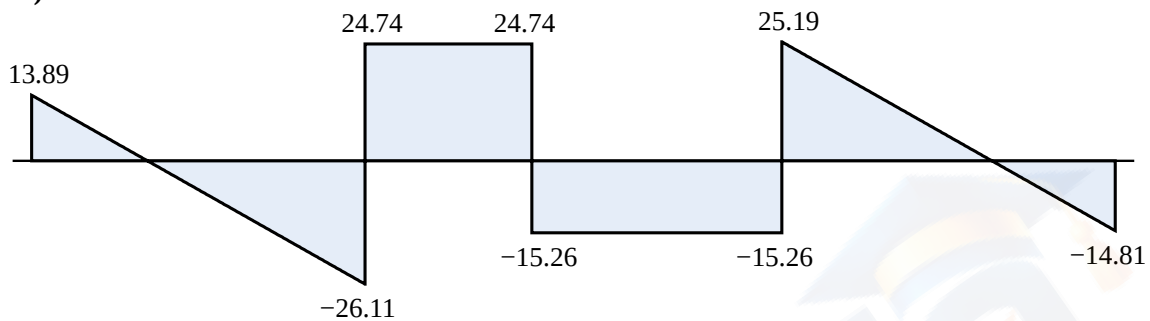
| Span     | Free reactions (left/right) kN | Correction (kN)            | Left end (kN) | Right end (kN) |
|----------|--------------------------------|----------------------------|---------------|----------------|
| AB (4 m) | 20 / 20                        | $(0 - 24.45)/4 = -6.11$    | 13.89         | 26.11          |
| BC (5 m) | 24 / 16                        | $(24.45 - 20.76)/5 = 0.74$ | 24.74         | 15.26          |
| CD (4 m) | 20 / 20                        | $(20.76 - 0)/4 = 5.19$     | 25.19         | 14.81          |

$R_A = 13.89 \text{ kN}$ ;  $R_B = 50.85 \text{ kN}$ ;  $R_C = 40.45 \text{ kN}$ ;  $R_D = 14.81 \text{ kN}$  ( $\Sigma R = 120 \text{ kN} = \text{total load} \checkmark$ )

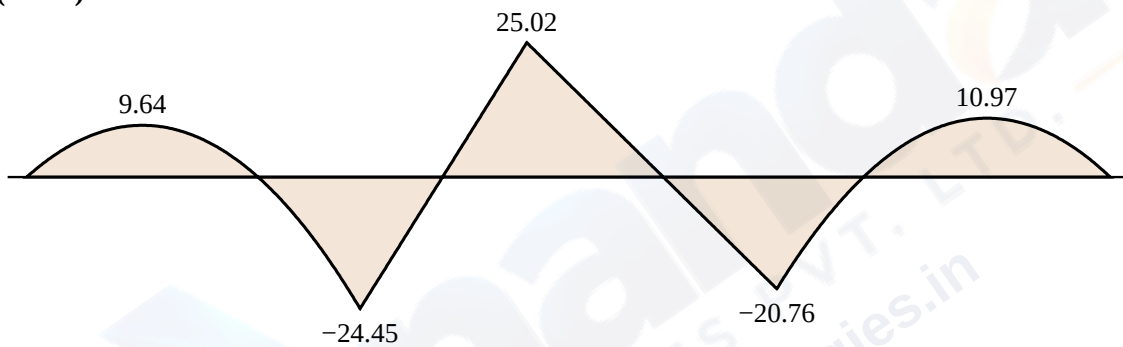
Maximum sagging moments — span AB: **9.64 kN·m** at 1.39 m from A; span BC: **25.02 kN·m** at 2 m from B; span CD: **10.97 kN·m** at 2.52 m from C.



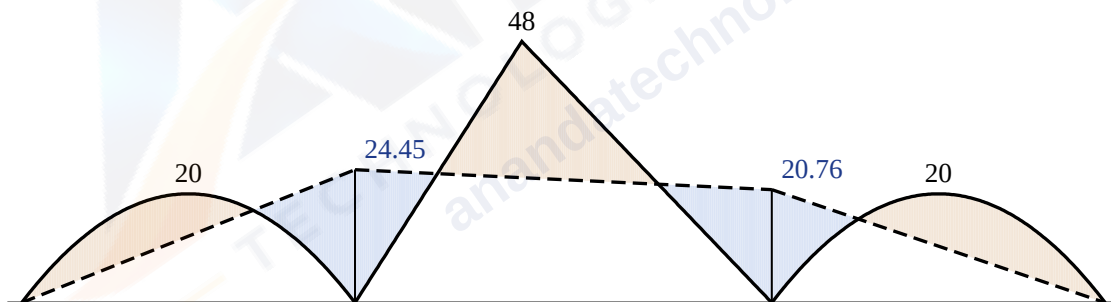
**SFD (kN)**



**BMD (kN·m)**



**BMD by superposition (kN·m): free BMD (solid) and support moment line (dashed)**



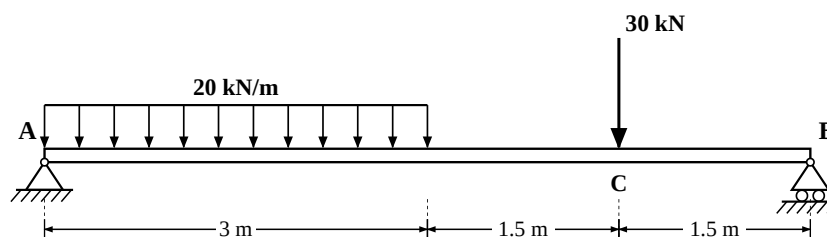
*SFD, BMD (net) and BMD by superposition (+ sagging, - hogging)*

✓ Check: support moments and reactions agree with an independent stiffness-matrix (and moment distribution) solution.

(6A&L terms 1 M, two equations 1 M, support moments 2 M, SFD 1 M, BMD 1 M)

## b) Macaulay's method — partial UDL, maximum deflection

6 M



**Step 1 — Reactions:**  $\Sigma M_A = 0$ :  $R_B \times 6 = 20 \times 3 \times 1.5 + 30 \times 4.5 \rightarrow R_B = 37.5 \text{ kN}$ ;  $R_A = (\text{total load}) - R_B = 52.5 \text{ kN}$

**Step 2 — Macaulay's equation:** Take A as origin and a section X-X at distance  $x$  from A in the last segment. (Terms after the vertical bar apply only when the bracket is positive.) A UDL that stops before the end is continued up to the section and an equal upward UDL is added (compensating load).

$$EI \frac{d^2y}{dx^2} = M_x = 52.5x - 10x^2 + 10[x - 3]^2 - 30[x - 4.5]$$

$$EI \frac{dy}{dx} = 26.25x^2 - 3.333x^3 + 3.333[x - 3]^3 - 15[x - 4.5]^2 + C_1 \quad \dots (1)$$

$$EI y = 8.75x^3 - 0.833x^4 + 0.833[x - 3]^4 - 5[x - 4.5]^3 + C_1x + C_2 \quad \dots (2)$$

**Step 3 — Boundary conditions:** At  $x = 0$ ,  $y = 0 \rightarrow$  (first segment of eq. 2)  $C_2 = 0$ . At  $x = 6 \text{ m}$ ,  $y = 0 \rightarrow$  (all terms of eq. 2)  $C_1 = -143.438$

### Step 4 — Results:

- Maximum deflection:** it occurs where slope = 0. Slope at  $x = 0 \text{ m}$  is negative and at  $x = 3 \text{ m}$  is positive, so the point lies between  $x = 0 \text{ m}$  and  $3 \text{ m}$ . Using the terms of eq. (1) valid in this segment:

$$EI \frac{dy}{dx} = -3.333x^3 + 26.25x^2 - 143.438 = 0 \rightarrow \text{solving (trial / calculator), } x = 2.958 \text{ m from A}$$

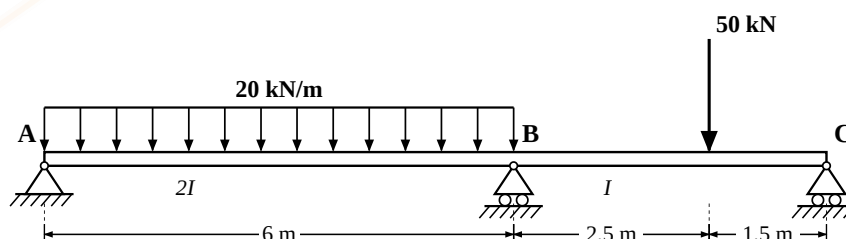
$$\text{Putting } x = 2.958 \text{ m in eq. (2): } EI y_{\max} = -261.62 \rightarrow y_{\max} = 261.62/EI$$

✓ Check: values agree with an independent stiffness-matrix analysis of the same beam.

(Reactions 1 M, equation with compensating UDL 1 M, constants 1 M, location 2 M, value 1 M)

## c) Moment distribution — two-span beam

6 M



**Step 1 — Fixed end moments (clockwise +, anticlockwise -):**

| Span | Left end formula             | $M_{\text{left}} \text{ (kN}\cdot\text{m)}$ | Right end formula | $M_{\text{right}} \text{ (kN}\cdot\text{m)}$ |
|------|------------------------------|---------------------------------------------|-------------------|----------------------------------------------|
| AB   | $wL^2/12 = 20 \times 6^2/12$ | -60                                         | $wL^2/12$         | +60                                          |

|    |                                              |               |                                              |              |
|----|----------------------------------------------|---------------|----------------------------------------------|--------------|
| BC | $Wab^2/L^2 = 50 \times 2.5 \times 1.5^2/4^2$ | <b>-17.58</b> | $Wa^2b/L^2 = 50 \times 2.5^2 \times 1.5/4^2$ | <b>+29.3</b> |
|----|----------------------------------------------|---------------|----------------------------------------------|--------------|

**Step 2 — Stiffness and distribution factors** ( $k = I/L$  for far end fixed or continuous;  $k = (3/4) \cdot I/L$  when the far end is simply supported):

| Joint | Member | Relative stiffness $k$      | $\Sigma k$ | Distribution factor              |
|-------|--------|-----------------------------|------------|----------------------------------|
| A     | AB     | $2I/6 = 0.3333I$            | 0.3333I    | 1 (simple end)                   |
| B     | BA     | $(3/4) \cdot 2I/6 = 0.25I$  | 0.4375I    | $0.25/0.4375 = \mathbf{0.571}$   |
|       | BC     | $(3/4) \cdot I/4 = 0.1875I$ |            | $0.1875/0.4375 = \mathbf{0.429}$ |
| C     | CB     | $I/4 = 0.25I$               | 0.25I      | 1 (simple end)                   |

**Step 3 — Moment distribution table** (carry-over factor  $1/2$ ; no carry-over to a simply supported end; the simple end is first released and half of the released moment is carried over):

| Step                 | A        | B            |               | C        |
|----------------------|----------|--------------|---------------|----------|
|                      | AB       | BA           | BC            | CB       |
| D.F.                 | 1        | 0.571        | 0.429         | 1        |
| FEM                  | -60      | 60           | -17.58        | 29.3     |
| Release simple end   | 60       |              |               | -29.3    |
| Carry over           |          | 30           | -14.65        |          |
| Balance              |          | -33.01       | -24.76        |          |
| <b>Final moments</b> | <b>0</b> | <b>56.99</b> | <b>-56.99</b> | <b>0</b> |

**Support moments:**  $M_A = 0 \text{ kN}\cdot\text{m}$ ;  $M_B = 56.99 \text{ kN}\cdot\text{m}$ ;  $M_C = 0 \text{ kN}\cdot\text{m}$  (hogging; final end moments at a joint are equal and opposite ✓).

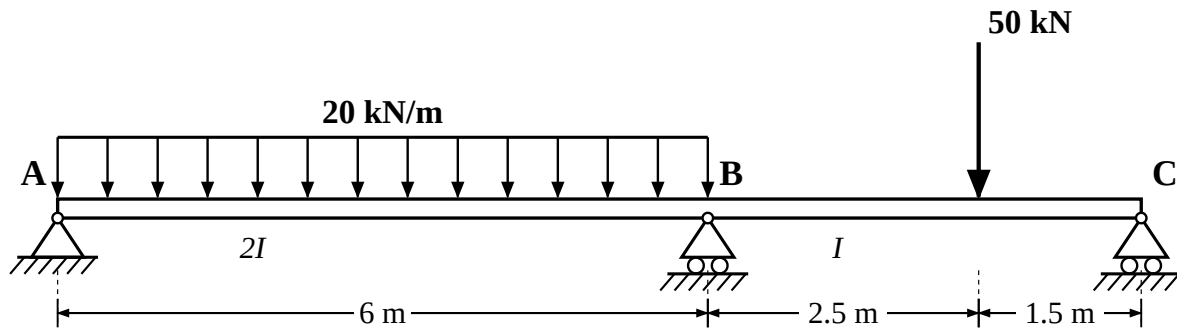
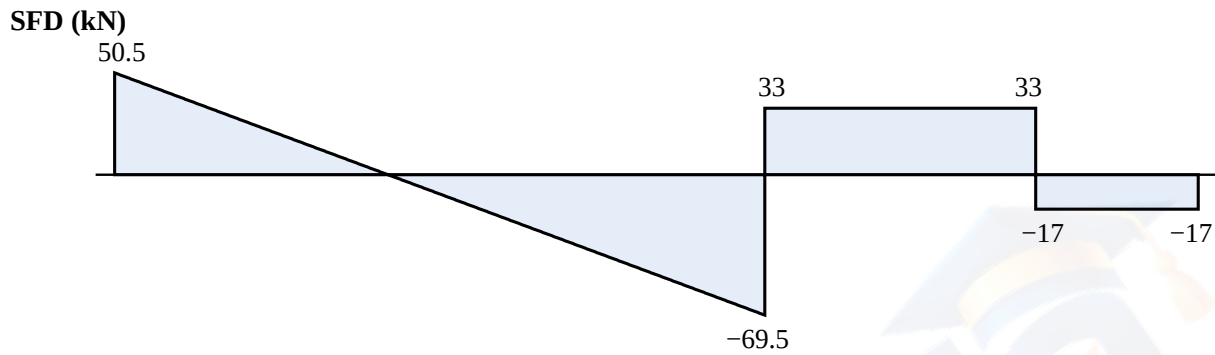
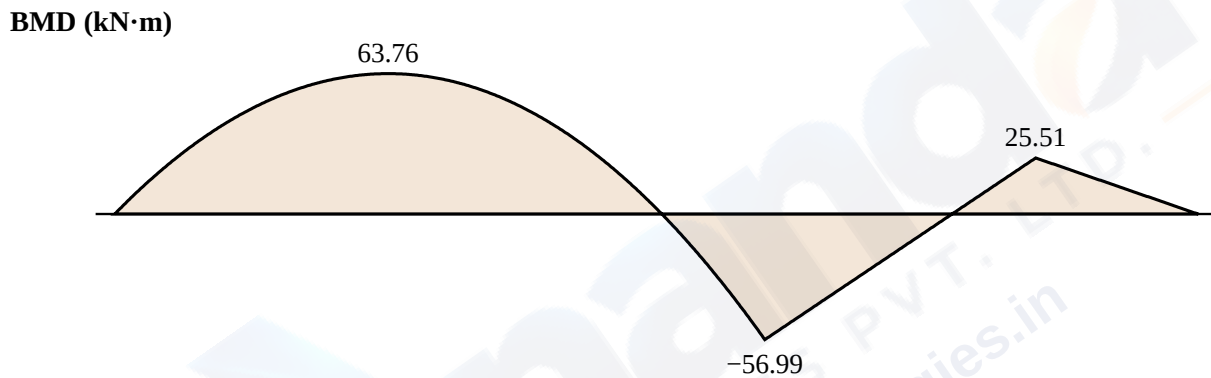
**Step 4 — Reactions and diagrams**

**Support reactions** (span-wise: free reaction  $\pm (M_{\text{left}} - M_{\text{right}})/L$ ):

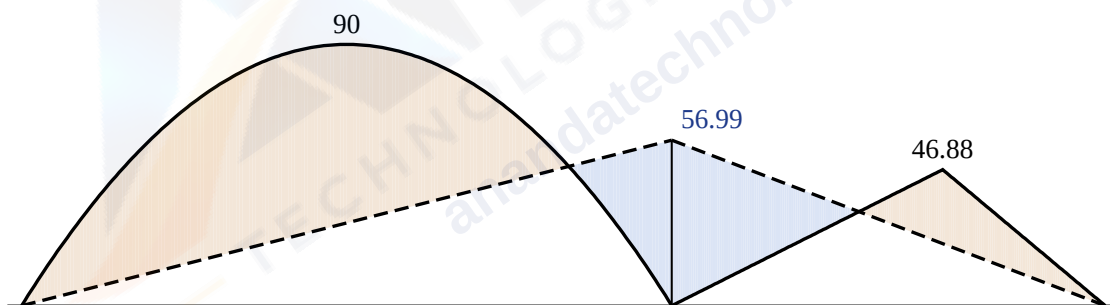
| Span     | Free reactions (left/right) kN | Correction (kN)         | Left end (kN) | Right end (kN) |
|----------|--------------------------------|-------------------------|---------------|----------------|
| AB (6 m) | 60 / 60                        | $(0 - 56.99)/6 = -9.5$  | 50.5          | 69.5           |
| BC (4 m) | 18.75 / 31.25                  | $(56.99 - 0)/4 = 14.25$ | 33            | 17             |

**$R_A = 50.5 \text{ kN}$ ;  $R_B = 102.49 \text{ kN}$ ;  $R_C = 17 \text{ kN}$**  ( $\Sigma R = 170 \text{ kN} = \text{total load}$  ✓)

Maximum sagging moments — span AB:  **$63.76 \text{ kN}\cdot\text{m}$**  at 2.52 m from A; span BC:  **$25.51 \text{ kN}\cdot\text{m}$**  at 2.5 m from B.



**BMD by superposition (kN·m): free BMD (solid) and support moment line (dashed)**



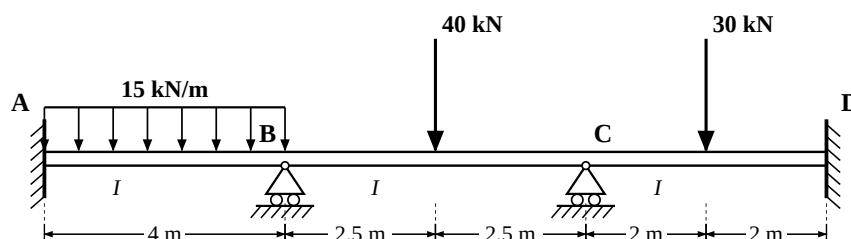
*SFD, BMD (net) and BMD by superposition (+ sagging, - hogging)*

✓ Check: support moments agree with the theorem of three moments and with a stiffness-matrix solve (difference < 0.01 kN·m).  
(FEM 1 M, DF 1 M, distribution 2 M, SFD and BMD 2 M)

## Q.6 Attempt any TWO of the following: (12 M)

### a) Moment distribution — beam fixed at both ends

6 M



#### Step 1 — Fixed end moments (clockwise +, anticlockwise -):

| Span | Left end formula             | $M_{\text{left}} \text{ (kN}\cdot\text{m)}$ | Right end formula | $M_{\text{right}} \text{ (kN}\cdot\text{m)}$ |
|------|------------------------------|---------------------------------------------|-------------------|----------------------------------------------|
| AB   | $wL^2/12 = 15 \times 4^2/12$ | -20                                         | $wL^2/12$         | +20                                          |
| BC   | $WL/8 = 40 \times 5/8$       | -25                                         | $WL/8$            | +25                                          |
| CD   | $WL/8 = 30 \times 4/8$       | -15                                         | $WL/8$            | +15                                          |

#### Step 2 — Stiffness and distribution factors ( $k = I/L$ for far end fixed or continuous; $k = (3/4) \cdot I/L$ when the far end is simply supported):

| Joint | Member | Relative stiffness $k$ | $\Sigma k$ | Distribution factor |
|-------|--------|------------------------|------------|---------------------|
| A     | AB     | $I/4 = 0.25I$          | $0.25I$    | 0 (fixed end)       |
| B     | BA     | $I/4 = 0.25I$          | $0.45I$    | $0.25/0.45 = 0.556$ |
|       | BC     | $I/5 = 0.2I$           |            | $0.2/0.45 = 0.444$  |
| C     | CB     | $I/5 = 0.2I$           | $0.45I$    | $0.2/0.45 = 0.444$  |
|       | CD     | $I/4 = 0.25I$          |            | $0.25/0.45 = 0.556$ |
| D     | DC     | $I/4 = 0.25I$          | $0.25I$    | 0 (fixed end)       |

#### Step 3 — Moment distribution table (carry-over factor $1/2$ ; no carry-over to a simply supported end):

| Step       | A    | B     |       | C     |       | D     |
|------------|------|-------|-------|-------|-------|-------|
|            | AB   | BA    | BC    | CB    | CD    | DC    |
| D.F.       | 0    | 0.556 | 0.444 | 0.444 | 0.556 | 0     |
| FEM        | -20  | 20    | -25   | 25    | -15   | 15    |
| Balance    |      | 2.78  | 2.22  | -4.44 | -5.56 |       |
| Carry over | 1.39 |       | -2.22 | 1.11  |       | -2.78 |
| Balance    |      | 1.23  | 0.99  | -0.49 | -0.62 |       |
| Carry over | 0.62 |       | -0.25 | 0.49  |       | -0.31 |
| Balance    |      | 0.14  | 0.11  | -0.22 | -0.27 |       |
| Carry over | 0.07 |       | -0.11 | 0.05  |       | -0.14 |
| Balance    |      | 0.06  | 0.05  | -0.02 | -0.03 |       |
| Carry over | 0.03 |       | -0.01 | 0.02  |       | -0.02 |

|               |        |       |        |       |        |       |
|---------------|--------|-------|--------|-------|--------|-------|
| Balance       |        | 0.01  | 0.01   | -0.01 | -0.01  |       |
| Final moments | -17.89 | 24.22 | -24.22 | 21.49 | -21.49 | 11.76 |

**Support moments:**  $M_A = 17.89 \text{ kN}\cdot\text{m}$ ;  $M_B = 24.22 \text{ kN}\cdot\text{m}$ ;  $M_C = 21.49 \text{ kN}\cdot\text{m}$ ;  $M_D = 11.76 \text{ kN}\cdot\text{m}$   
(hogging; final end moments at a joint are equal and opposite ✓).

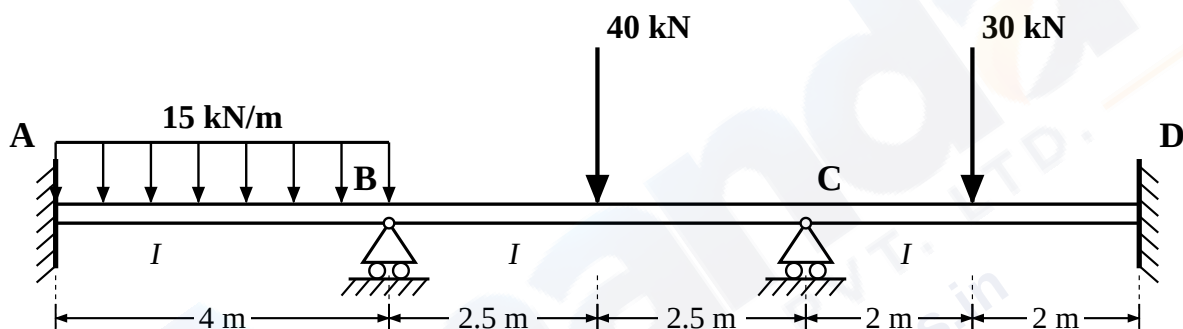
#### Step 4 — Reactions and diagrams

**Support reactions** (span-wise: free reaction  $\pm (M_{\text{left}} - M_{\text{right}})/L$ ):

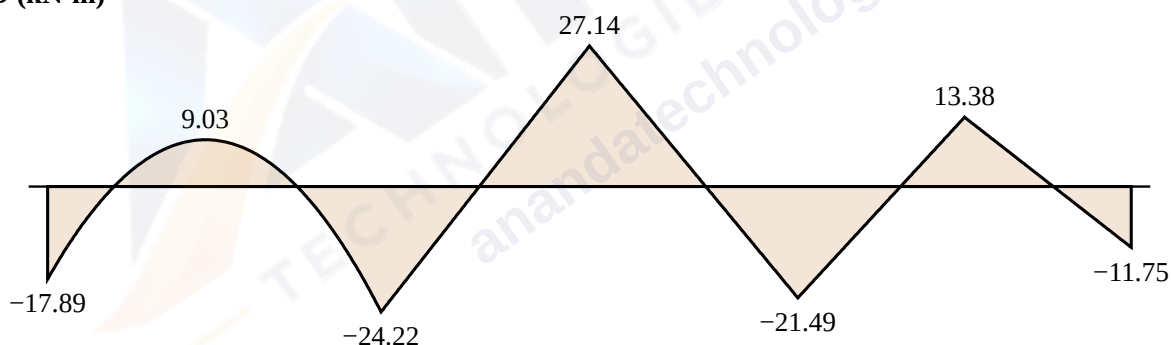
| Span     | Free reactions (left/right) kN | Correction (kN)             | Left end (kN) | Right end (kN) |
|----------|--------------------------------|-----------------------------|---------------|----------------|
| AB (4 m) | 30 / 30                        | $(17.89 - 24.22)/4 = -1.58$ | 28.42         | 31.58          |
| BC (5 m) | 20 / 20                        | $(24.22 - 21.49)/5 = 0.55$  | 20.55         | 19.45          |
| CD (4 m) | 15 / 15                        | $(21.49 - 11.76)/4 = 2.43$  | 17.43         | 12.57          |

$R_A = 28.42 \text{ kN}$ ;  $R_B = 52.13 \text{ kN}$ ;  $R_C = 36.89 \text{ kN}$ ;  $R_D = 12.57 \text{ kN}$  ( $\Sigma R = 130 \text{ kN} = \text{total load}$  ✓)

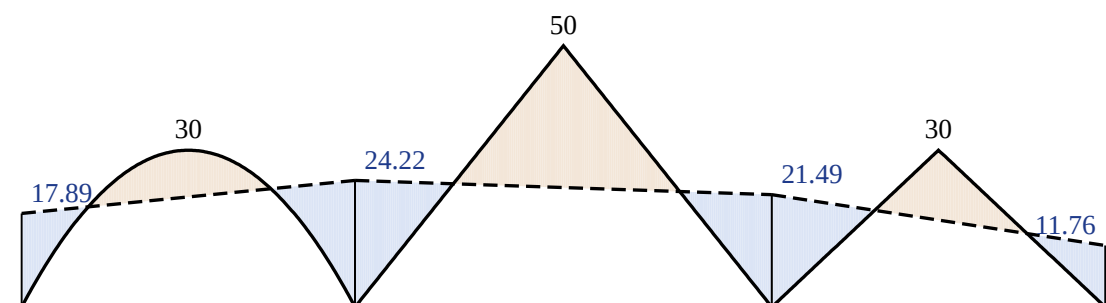
Maximum sagging moments — span AB: **9.03 kN·m** at 1.89 m from A; span BC: **27.14 kN·m** at 2.5 m from B; span CD: **13.38 kN·m** at 2 m from C.



**BMD (kN·m)**



**BMD by superposition (kN·m): free BMD (solid) and support moment line (dashed)**

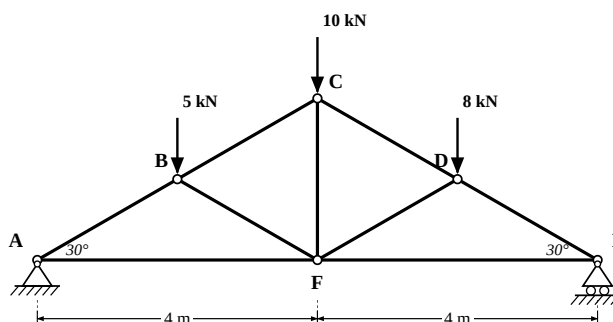


*BMD (net) and BMD by superposition (+ sagging, - hogging)*

✓ Check: support moments agree with the theorem of three moments and with a stiffness-matrix solve (difference < 0.01 kN·m).

## b) Roof truss — method of joints

6 M



**Check:** number of joints  $j = 6$ , members  $m = 9$ ;  $2j - 3 = 9 = m \rightarrow$  **perfect frame.**

**Support reactions:**  $\Sigma M_A = 0$ :  $R_E \times 8 = 5 \times 2 + 10 \times 4 + 8 \times 6 \rightarrow R_E = 12.25 \text{ kN } (\uparrow)$ ;  $\Sigma F_y = 0$ :  $R_A = 23 - 12.25 = 10.75 \text{ kN } (\uparrow)$ ;  $\Sigma F_x = 0$ :  $H_A = 0$  (all loads vertical).

All unknown member forces are first assumed **tensile** (acting away from the joint). A negative answer means the member is in **compression**.

**Joint A** (unknowns:  $F_{AB}$ ,  $F_{AF}$ ):

$$\Sigma F_x = 0: F_{AB} \cos 30^\circ + F_{AF} = 0$$

$$\Sigma F_y = 0: F_{AB} \sin 30^\circ + 10.75 = 0$$

$$\rightarrow F_{AB} = -21.5 \text{ kN (C)}, F_{AF} = 18.62 \text{ kN (T)}$$

**Joint B** (unknowns:  $F_{BC}$ ,  $F_{BF}$ ):

$$\Sigma F_x = 0: -(-21.5) \cos 30^\circ + F_{BC} \cos 30^\circ + F_{BF} \cos 30^\circ = 0$$

$$\Sigma F_y = 0: -(-21.5) \sin 30^\circ + F_{BC} \sin 30^\circ - F_{BF} \sin 30^\circ - 5 = 0$$

$$\rightarrow F_{BC} = -16.5 \text{ kN (C)}, F_{BF} = -5 \text{ kN (C)}$$

**Joint C** (unknowns:  $F_{CD}$ ,  $F_{CF}$ ):

$$\Sigma F_x = 0: -(-16.5) \cos 30^\circ + F_{CD} \cos 30^\circ = 0$$

$$\Sigma F_y = 0: -(-16.5) \sin 30^\circ - F_{CD} \sin 30^\circ - F_{CF} - 10 = 0$$

$$\rightarrow F_{CD} = -16.5 \text{ kN (C)}, F_{CF} = 6.5 \text{ kN (T)}$$

**Joint F** (unknowns:  $F_{FE}$ ,  $F_{FD}$ ):

$$\Sigma F_x = 0: -(18.62) + F_{FE} - (-5) \cos 30^\circ + F_{FD} \cos 30^\circ = 0$$

$$\Sigma F_y = 0: (-5) \sin 30^\circ + (6.5) + F_{FD} \sin 30^\circ = 0$$

$$\rightarrow F_{FE} = 21.22 \text{ kN (T)}, F_{FD} = -8 \text{ kN (C)}$$

**Joint D** (unknowns:  $F_{DE}$ ):

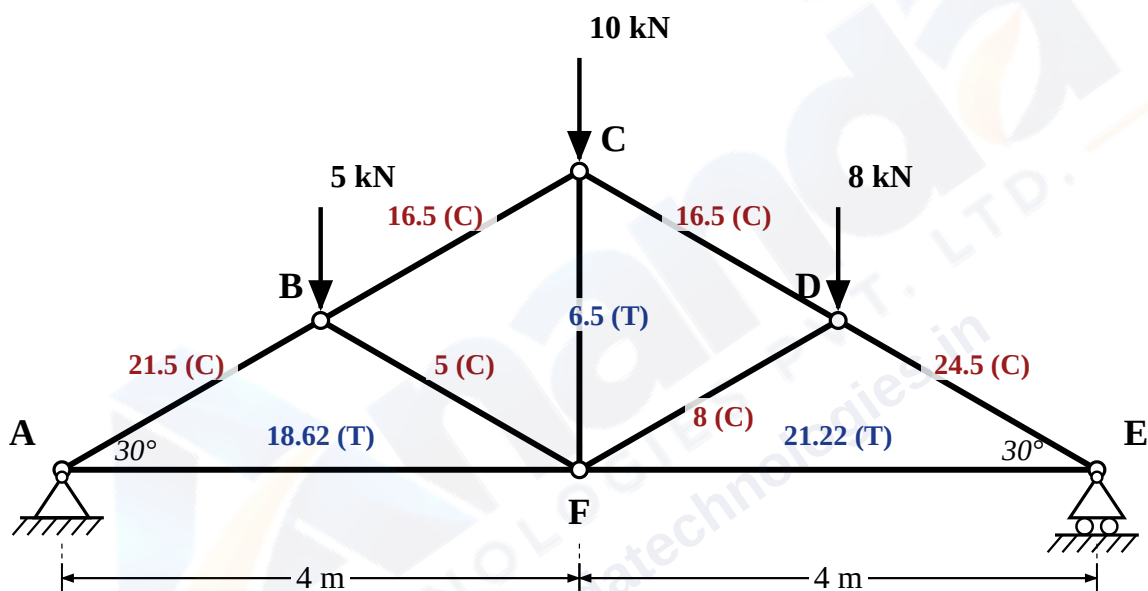
$$\Sigma F_x = 0: -(-16.5) \cos 30^\circ + F_{DE} \cos 30^\circ - (-8) \cos 30^\circ = 0$$

$$\Sigma F_y = 0: (-16.5) \sin 30^\circ - F_{DE} \sin 30^\circ - (-8) \sin 30^\circ - 8 = 0$$

$$\rightarrow F_{DE} = -24.5 \text{ kN (C)}$$

## Result table

| Member | Force (kN) | Nature      |
|--------|------------|-------------|
| AB     | 21.5       | Compression |
| BC     | 16.5       | Compression |
| CD     | 16.5       | Compression |
| DE     | 24.5       | Compression |
| AF     | 18.62      | Tension     |
| FE     | 21.22      | Tension     |
| BF     | 5          | Compression |
| CF     | 6.5        | Tension     |
| DF     | 8          | Compression |



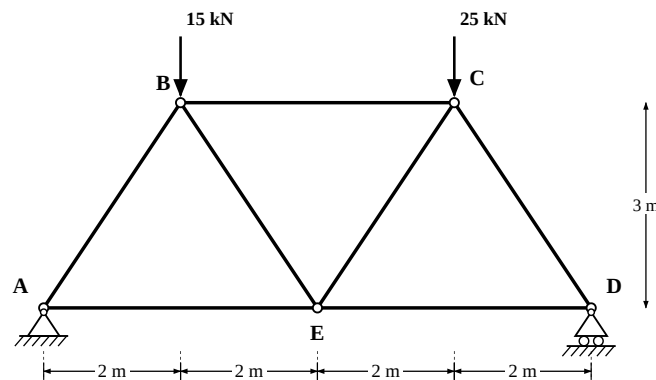
Member forces (T = tension, C = compression)

✓ Check: all member forces agree with a full matrix solution of the 2j joint-equilibrium equations.

(Reactions 1 M, joint equations 3 M, results with nature 2 M)

### c) Truss — method of sections

6 M



**Support reactions:**  $\Sigma M_A = 0$ :  $R_D \times 8 = 15 \times 2 + 25 \times 6 \rightarrow R_D = 22.5 \text{ kN } (\uparrow)$ ;  $\Sigma F_y = 0$ :  $R_A = 40 - 22.5 = 17.5 \text{ kN } (\uparrow)$ ;  $\Sigma F_x = 0$ :  $H_A = 0$  (all loads vertical).

Draw section line 1–1 cutting members CB, CE, DE and consider the equilibrium of the part containing joints C, D. Forces in cut members are assumed tensile (away from the part).

**Force in CB:** take moments about E (anticlockwise +). Perpendicular distance of CB from this point = 3 m.

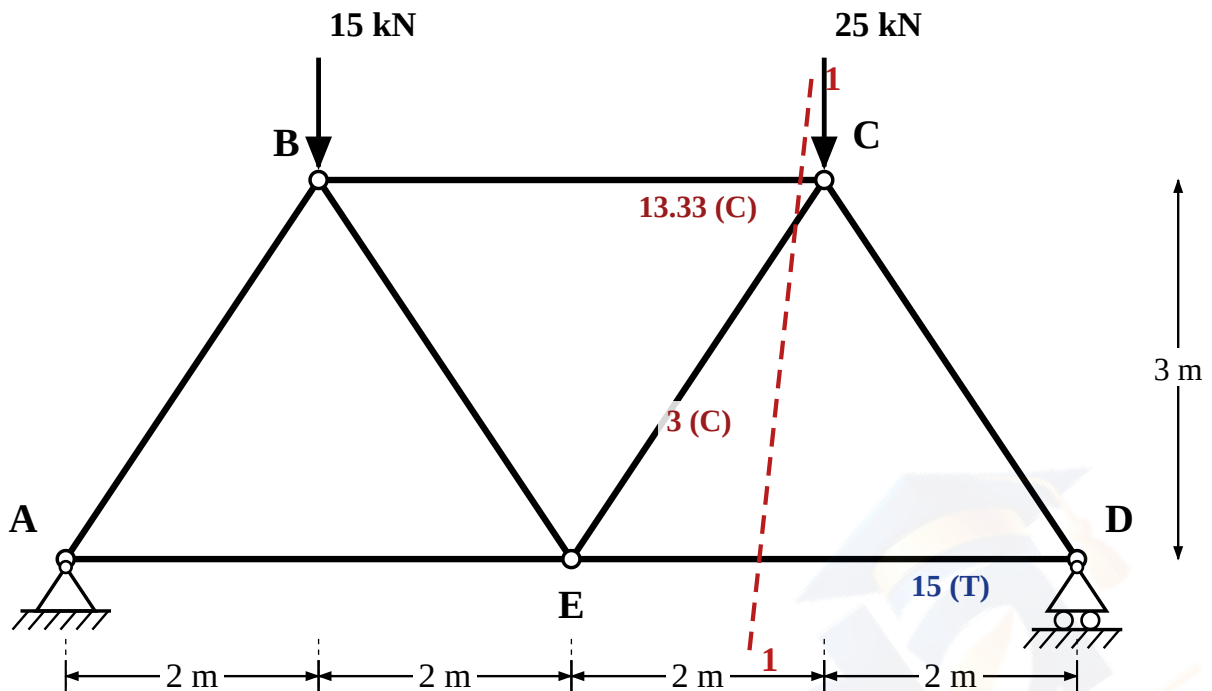
$$\Sigma M = 0: -25 \times 2 + 22.5 \times 4 + F_{CB} \times 3 = 0 \rightarrow F_{CB} = -13.33 \text{ kN (C)}$$

**Force in CE:** members CB and DE are horizontal (parallel), so resolve vertically:  $\Sigma F_y = 0$ :  $-25 + 22.5 - F_{CE} \sin 56.31^\circ = 0 \rightarrow F_{CE} = -3 \text{ kN (C)}$

**Force in DE:** take moments about C (anticlockwise +). Perpendicular distance of DE from this point = 3 m.

$$\Sigma M = 0: 22.5 \times 2 - F_{DE} \times 3 = 0 \rightarrow F_{DE} = 15 \text{ kN (T)}$$

| Member | Force (kN) | Nature      |
|--------|------------|-------------|
| CB     | 13.33      | Compression |
| CE     | 3          | Compression |
| DE     | 15         | Tension     |



*Section 1–1 and forces in the cut members*

✓ Check: values agree with the full matrix (method of joints) solution of the truss.

*(Reactions 1 M, section 1 M, each force with nature 4 M)*

**Note:** This is a **model question paper prepared by Ananda Technologies** only for practice and exam preparation, based on the pattern of previous MSBTE question papers and the K-scheme syllabus. It is **not an official MSBTE paper**, and it does **not** mean that these questions will come in the exam. Answers follow standard textbook points; marks shown are indicative. Free for students — [anandatechnologies.in](http://anandatechnologies.in)